CSCA67 Tutorial, Week 7

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Adapted from
A. Bretscher, *Implication and Direct Proofs worksheet*,
A. Bretscher, *Introduction to Proofs Lecture Notes* &
A. Bretscher, *CSCA65 Lecture Notes: Formal Languages - Logic*

1 Formal Logic

A statement (also known as a proposition) is a sentence that can be evaluated to be true or false.

Q: *Which of the following are statements?*

(1) $2 \times 4 = 8$	statement (true)
(2) $5 = 9$	statement (false)
(3) It will snow this afternoon.	statement (value is unknown, but must evaluate to true or false)
(4) If I am happy, then I am not happy.	statement (false)
(5) Give the definition of a statement.	not a statement
(6) This statement is false.	not a statement (cannot be either true or false - if it is true, then it is false; if it is false, then it is true)
(7) This statement is true.	not a statement (value cannot be determined, could be evaluated as either true or false)

We represent statements using symbols (which are often arabic or Greek letters).
For example, we can let p represent the statement "It is raining," and q represent the statement "I have an umbrella."

We can build more complex statements (known as "compound statements") by combining statements using any of the following connectives, or operators.

Symbol	Meaning	Example		Precedence
$\neg$	"not"	$\neg p$	It is *not* raining.	highest
$\wedge$	"and"	$p \wedge q$	It is raining *and* I have an umbrella.	
$\vee$	"or"	$p \vee q$	It is raining *or* I have an umbrella.	
$\rightarrow$	"implies"	$p \rightarrow q$	*If* it is raining, *then* I have an umbrella.	
$\leftrightarrow$	"if and only if"	$p \leftrightarrow q$	It is raining *if and only if* I have an umbrella.	lowest precedence

We may also use parentheses to group statements and connectives. When parentheses are omitted, the connectives are applied according to the precedence rules (also called the "order of operations").

For example, when $\neg A \wedge B$ is parenthesized, it becomes $\neg(A) \wedge B$, since $\neg$ has higher precedence than $\wedge$.

Note that this is entirely different from $\neg(A \wedge B)$.
Likewise, when $A \wedge B \rightarrow \neg C \vee D \wedge E$ is parenthesized, according to the precedence rules, it becomes $(A \wedge B) \rightarrow ((\neg C) \vee (D \wedge E))$.

When an operator is repeated, sub-expressions are usually grouped to the right. For example, when $A \rightarrow B \rightarrow C$ is parenthesized, it becomes $A \rightarrow (B \rightarrow C)$.

Given the statements
p : **"You are in Seoul."**
q : **"You are in Kwangju."**
r : **"You are in South Korea."**

Q: *Translate the following statement into formal logic:* *"If you are not in South Korea, then you are not in Seoul or Kwangju."*

We start with the most general statement: from the language "if…then…", we know that the statement is an implication, with "you are not in South Korea" as the antecedent (first half) and "you are not in Seoul or Kwangju" as the consequent (second half).

"You are not in South Korea" is the negation of r, and "you are not in Seoul or Kwangju" is the negation of "you are in Seoul or you are in Kwangju".

"You are in Seoul or you are in Kwangju" is the disjunction ("or") of p and q.

Thus, in formal logic, our statement is $r \rightarrow \neg(p \vee q)$.

Alternatively, because of the ambiguity of the English language, we may consider "you are not in Seoul or Kwangju" to be the conjunction ("and") of "you are not in Seoul" and "you are not in Kwangju", which are, respectively, the negation of p and q. Then our statement would be $r \rightarrow \neg p \wedge \neg q$.

Q: *Translate the following formal statement into everyday English:* $q \rightarrow (r \wedge \neg p)$

Here, we start with the most specific statement: $\neg p$ is the negation of "you are in Seoul", which we express in English as "you are not in Seoul".

Then, $r \wedge \neg p$ is the conjunction ("and") of "you are not in Seoul" and "you are in South Korea", which we express as "you are in South Korea and you are not in Seoul".

Finally, we translate the implication to "if…then…", with "you are in Kwangju" as the antecedent (first half) and "you are in South Korea and you are not in Seoul" as the consequent (second half).

Thus, in English, our statement is "If you are in Kwangju, then you are in South Korea and you are not in Seoul".

Because of the ambiguity of the English language, there are several equivalent ways to structure this sentence. For example, we might say "If you are in Kwangju, then you are in South Korea *but* you are not in Seoul", or simply "If you are in Kwangju, then you are in South Korea but not in Seoul".

1.1 Truth Tables

A TRUTH TABLE is a table showing all possible truth values for a statement, depending upon the statements that make it up.

When we construct a truth table for a statement A containing operator(s) and constituent statements A_1, A_2, …, we show the possible truth values of A_1, A_2, … on the left side of the table, and the corresponding truth values for A on the right side of the table.
Each row of the truth table represents the truth value A given the truth values of A_1, A_2, … in that row.

The most basic truth table is the truth table for the single statement p, shown below. p can take either a true (T) or false (F) value.

NOTE that the order of the rows in a truth table is not significant, although a conventional ordering can make the truth table easier to read.

p
T
F

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

A TRUTH TABLE is particularly useful for assessing the possible truth values of a complex statement (i.e., one which contains many operators and/or constituent statements). When constructing such a truth table, we break the statement into smaller clauses and assess the truth value of each, before combining them into the original statement.
For example, here is a (partial) truth table for $(\neg r \vee q) \wedge p$:

r	q	p	$\neg r$	$\neg r \vee q$	$(\neg r \vee q) \wedge p$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	F
	⋮		⋮	⋮	⋮

Given the statements

p**: "Andy is hungry."**

q**: "The refrigerator is empty."**

r**: "Andy is mad."**

Q: *Construct a truth table for the following statement: "If Andy is hungry and the refrigerator is empty, then Andy is mad."*

First, we must translate this statement into formal logic. From the language "if…then…", we know that the statement is an implication, with "Andy is hungry and the refrigerator is empty" as the antecedent and "Andy is mad" as the consequent. "Andy is hungry and the refrigerator is empty" is the conjunction of p and q.
Thus, in formal logic, our statement is $p \wedge q \rightarrow r$.

To build our truth table, we begin with the smallest constituent statements, p, q, and r, and detemine all possible combinations of truth values for these.

Then, we identify the next smallest sub-statement, based on precedence rules and parentheses - in this case, it is $p \wedge q$. We determine the truth values of this statement using the truth values of the statements that make it up - in this case, p and q.

We continue doing this until we have truth values for all sub-statements within our original statement. Finally, we use these truth values to determine the truth values of the original statement.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$\Rightarrow$

p	q	r	$p \wedge q$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

$\Rightarrow$

p	q	r	$p \wedge q$	$p \wedge q \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Suppose that this statement is true, and that Andy is not mad and the refrigerator is empty.

Q: *Is Andy hungry?*

Each of our assumptions assigns a truth value to a statement in our truth table:

- "this statement is true" $\Rightarrow p \wedge q \rightarrow r$ is true
- "Andy is not mad" $\Rightarrow r$ is false
- "the refrigerator is empty" $\Rightarrow q$ is true

We are then being asked to find the truth value of p, "Andy is hungry".

We find the row of our truth table which corresponds to the truth values we know.

p	q	r	...	$p \wedge q \rightarrow r$
T	T	T		T
T	T	F		F
T	F	T		T
T	F	F		T
F	T	T	...	T
F	T	F		T
F	F	T		T
F	F	F		T

We can see that p is false in this row. Thus, Andy is *not* hungry.

1.2 Logical Equivalence

Two statements are LOGICALLY EQUIVALENT if they have the same truth table - that is, given the same combination of truth values for their constituent statements, they both have the same truth value.

For example, using truth tables, we can demonstrate that p and $\neg\neg p$ are logically equivalent:

p
T
F

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

Likewise, we can demonstrate that $\neg y \wedge (y \vee x)$ and $\neg y \wedge x$ are logically equivalent:

x	y	$\neg y$	$y \vee x$	$\neg y \wedge (y \vee x)$	$\neg y \wedge x$
T	T	F	T	F	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	F

Q: *Show that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$ using truth tables.*

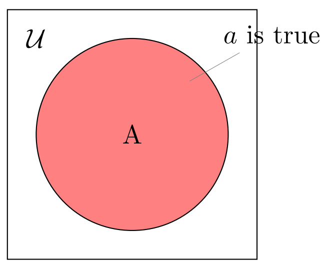
p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\neg(p \wedge q)$ and $\neg p \vee \neg q$ have the same truth table - thus, they are logically equivalent statements.

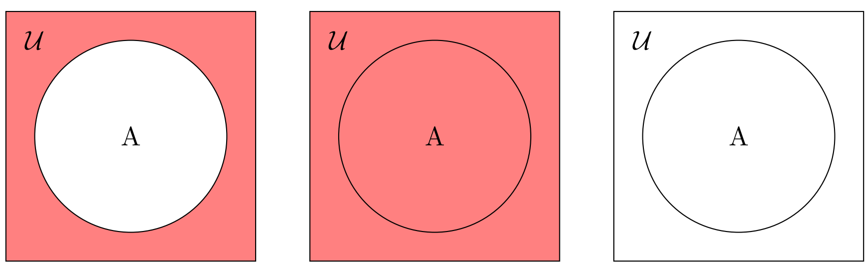
1.3 Using Venn Diagrams

We can consider any statement a to be the statement "$x \in A$", where A is a set and x is an element.

Then, if we draw a Venn diagram containing A, a is true at every location in the diagram where an element x at that location is in A. a is false everywhere else.



Q: *Shade the region(s) where each of the following is true: $\neg a$, $\neg a \vee a$, $\neg a \wedge a$.*

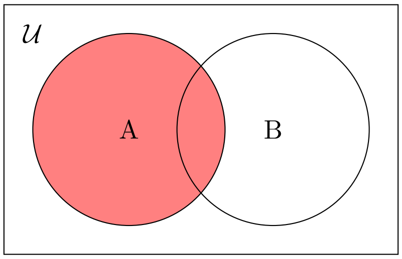


$\neg(x \in A)$ $\neg(x \in A) \vee (x \in A)$ $\neg(x \in A) \wedge (x \in A)$

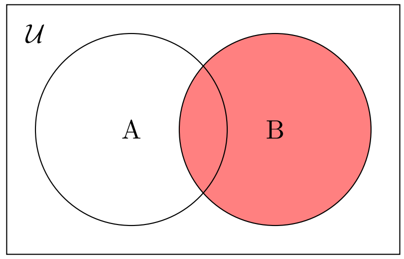
Notice that $\neg a \vee a$ is true for every region in the diagram. This is because, regardless of the truth value of a (or of its constituent statements), $\neg a \vee a$ is always true. This type of statement is known as a *tautology*.

Notice also that $\neg a \wedge a$ is false for every region in the diagram. This is because, regardless of the truth value of a (or of its constituent statements), $\neg a \wedge a$ is always false. This type of statement is known as a *contradiction*.

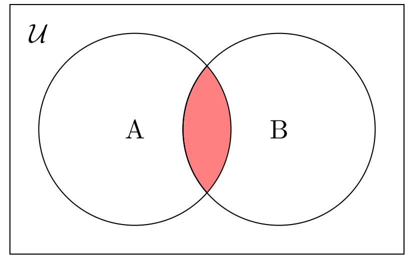
We follow the same process to represent a statement with multiple constituent statements.
For example, to create a Venn diagram representing $a \wedge b$, we let a be the statement "$x \in A$" and b be the statement "$x \in B$", where A and B are sets and x is an element. Then



$x \in A$

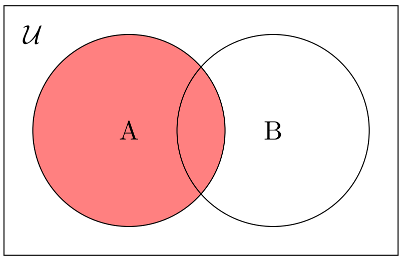


$x \in B$

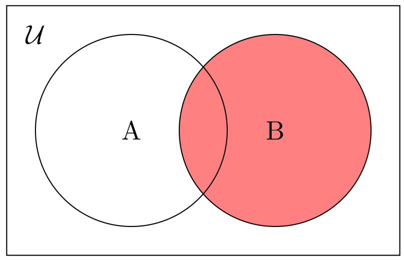


$(x \in A) \wedge (x \in B)$

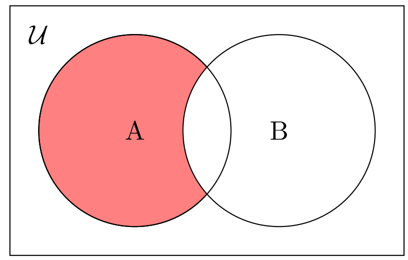
Q: *Shade the region(s) where* $a \to b$ *is false.*



$x \in A$



$x \in B$



$x \in A$ but $x \notin B$

Notice that this region is the region in which a is true but b is false. Everywhere else (that is, for every other combination of truth values for a and b), the implication is true.

2 Additional practice problems

Q: Which of the following are valid propositions?

- $2 + 2 = 4$
- $2 + 3 = 7$
- If it is sunny tomorrow, I will go to the beach.
- What is going on?
- Stop at the red light.

Q: Which of the following propositions are true?

- If a, b, c are two sides and the hypotenuse of a triangle, then $a^2 + b^2 = c^2$.
- If $2 + 2 = 4$ then pigs can fly.
- If pigs can fly then pigs can get sunburned.
- If $2 + 2 = 5$ then $2 + 2 = 4$.

Q: Use truth tables to prove the following *distributive* properties:

$p \wedge (q \vee r)$ is logically equivalent to $(p \wedge q) \vee (p \wedge r)$.

$p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$.

Q: Use truth tables to prove the following *associative* properties:

$p \vee (q \vee r)$ is logically equivalent to $(p \vee q) \vee r$.

$p \wedge (q \wedge r)$ is logically equivalent to $(p \wedge q) \wedge r$.

There are 3 boxes A, B, C. Exactly one contains gold.
Each box has a message on top, but only one of the messages is true.
Box A: *Gold is not in this box.* Box B: *Gold is not in this box.* Box C: *Gold is in box A.*
Q: Use implication to determine which box the gold is in.

Hint: Start with $\{p$: Gold is in box A, q: Gold is in box B, r: Gold is in box C$\}$, and rewrite the messages using these propositions.

Q: Construct a truth table for the following compound statement:
$\neg(\neg(X \vee \neg Y \vee \neg Z) \vee (\neg X \wedge Y \wedge \neg Z)) \vee (\neg X \vee (\neg Y \wedge Z))$

Q: Use a truth table to prove that the statement $((p \vee q) \wedge (\neg p)) \to q$ is always true, no matter what p and q are.